New properties of hypergeometric series derivable from Feynman integrals. I. Transformation and reduction formulae

1987 J. Phys. A: Math. Gen. 20 4109

(http://iopscience.iop.org/0305-4470/20/13/019)

J. Phys. A: Math. Gen. **20** (1987) 4109-4117. Printed in the UK

New properties of hypergeometric series derivable from Feynman integrals: I. Transformation and reduction formulae

A A Inayat-Hussain
Department of Physics, University of Western Australia, Nedlands, WA 6009, Australia

Received 25 June 1986, in final form 16 December 1986

Abstract. A study in the statistical mechanics of phase transitions has involved certain seemingly simple Feynman integrals which have yielded surprisingly significant mathematical results. The integrals enable the derivation of new transformation, summation and reduction formulae for some single and multiple variable hypergeometric series. The main result reported in this paper is a transformation formula, a special case of which is a summation formula for a symmetric combination of two non-terminating Clausen's hypergeometric series of unit argument.

1. Introduction

Single and multiple variable hypergeometric functions provide a convenient and powerful mode of mathematical description for a wide variety of problems arising in the physical sciences, engineering and statistics (see, e.g., Srivastava and Karlsson 1985, § 1.7 and references therein). The special functions (see Erdélyi *et al* 1953) are mostly cases of hypergeometric functions, which are in turn cases of the more general H function of one or more variables (Srivastava *et al* 1982, Srivastava and Goyal 1985); the so-called generalised hypergeometric function (Slater 1966), the Kampé de Fériet function (Kampé de Fériet 1921, Appell and Kampé de Fériet 1926) and the Lauricella functions (Lauricella 1893, Exton 1978, ch 2) have been studied in particular and many useful properties such as transformation, summation and reduction formulae are established.

Recently Niukkanen (1983, 1984), in an attempt to provide a unification of the types of functions which arise in multi-centre calculations in quantum chemistry, has re-introduced a certain multivariable hypergeometric function and studied some of its properties. Srivastava (1985a, b) has clarified this unification.

New properties to be discussed here are summation, transformation and reduction formulae for some single and multiple variable hypergeometric functions, including the function recently studied by Niukkanen and Srivastava. In the following paper (Inayat-Hussain 1987a, hereafter referred to as II) certain analytic continuation formulae and a new generalisation of the H function are presented.

Our new results are obtained from evaluating in two ways certain Feynman integrals which arise in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions (Ma 1976, Inayat-Hussain and Buckingham 1986a). All these integrals are listed in a table in II, together with their convergence conditions.

0305-4470/87/134109+09\$02.50

In this paper, we are concerned only with the d-dimensional integrals:

$$f \equiv f(\alpha, \beta, \nu, d) = (2\pi)^{-d} \int \mathrm{d}\boldsymbol{p}\, |\boldsymbol{p}|^{-\alpha} |\boldsymbol{p}\cdot\mathbf{1}|^{\nu} |\boldsymbol{p}+\mathbf{1}|^{-\beta}$$

and

$$f_{1/2} \equiv f_{1/2}(\alpha, \beta, \nu, d) = (2\pi)^{-d} \int_R \mathrm{d}\boldsymbol{p}\, |\boldsymbol{p}|^{-\alpha} |\boldsymbol{p}\cdot\mathbf{1}|^{\nu} |\boldsymbol{p}+\mathbf{1}|^{-\beta}$$

where $\alpha < d+\nu < \alpha+\beta$ and $\nu > -1$ and the region R is the half-space in which $\boldsymbol{p}\cdot\mathbf{1} \geq 0$, $\mathbf{1}$ being an arbitrary unit vector. For f we also require that $\beta < d$.

In § 2 we recall and summarise the notation and definitions for some hypergeometric functions. In § 3 the integral $f_{1/2}$ is used to derive a reduction formula for an infinite series of ${}_4F_3(1)$ into a single ${}_3F_2(1)$. In conjunction with the integral f, the reduction formula is then used in § 4 to derive a transformation of a ${}_4F_3(1)$ into a series of ${}_3F_2(1)$. A special case of this transformation is a summation formula, equation (12) below, for a symmetric combination of two ${}_3F_2(1)$.

Finally, in § 5, we conclude and mention some open problems in the theory of single and multiple variable hypergeometric functions.

An appendix contains the details of the two evaluations of the integral f.

2. Notation and definitions

The Pochhammer symbol $(a)_k$ is given by $\Gamma(a+k)/\Gamma(a)$ where $\Gamma(z)$ is the gamma function.

(i) Generalised hypergeometric function ${}_pF_q(z)$

$${}_pF_q\left[{a_1,\ldots,a_p;\atop b_1,\ldots,b_q;}\, z\right] = \sum_{n=0}^{\infty} \frac{\Pi_{i=1}^{p}(a_i)_n z^n}{\Pi_{j=1}^{q}(b_j)_n n!}$$
$$= {}_pF_q[a_1,\ldots,a_p; b_1,\ldots,b_q; z]. \tag{1}$$

(ii) Wright's function ${}_p\Psi_q(z)$

$${}_p\Psi_q\left[{(\alpha_1, A_1),\ldots,(\alpha_p, A_p);\atop (\beta_1, B_1),\ldots,(\beta_q, B_q);}\, z\right] = \sum_{n=0}^{\infty} \frac{\Pi_{i=1}^{p}\Gamma(\alpha_i + A_i n) z^n}{\Pi_{j=1}^{q}\Gamma(\beta_j + B_j n) n!}. \tag{2}$$

(iii) Kampé de Fériet function

$$F^{p:\,q;\,k}_{l:\,m;\,n}\left[{a_1,\ldots,a_p : b_1,\ldots,b_q ; c_1,\ldots,c_k ;\atop \alpha_1,\ldots,\alpha_l : \beta_1,\ldots,\beta_m ; \gamma_1,\ldots,\gamma_n ;}\, z_1, z_2\right] = \sum_{r,s=0}^{\infty} \frac{\Pi_{i=1}^{p}(a_i)_{r+s}\,\Pi_{i=1}^{q}(b_i)_r\,\Pi_{i=1}^{k}(c_i)_s z_1^r z_2^s}{\Pi_{i=1}^{l}(\alpha_i)_{r+s}\,\Pi_{i=1}^{m}(\beta_i)_r\,\Pi_{i=1}^{n}(\gamma_i)_s\, r!\, s!}. \tag{3}$$

(iv) Generalised Lauricella function of n variables (in a notation modified slightly from that first introduced by Srivastava and Daoust (1969))

$$F^{A:\,B^{(1)};\ldots;\,B^{(n)}}_{C:\,D^{(1)};\ldots;\,D^{(n)}}\left[{[\boldsymbol{a}:\theta^{(1)},\ldots,\theta^{(n)}]:[\boldsymbol{b}^{(1)}:\phi^{(1)}];\ldots;[\boldsymbol{b}^{(n)}:\phi^{(n)}];\atop [\boldsymbol{c}:\Psi^{(1)},\ldots,\Psi^{(n)}]:[\boldsymbol{d}^{(1)}:\delta^{(1)}];\ldots;[\boldsymbol{d}^{(n)}:\delta^{(n)}];}\, z_1,\ldots,z_n\right]$$
$$= \frac{\Pi_{j=1}^{C}\Gamma(c_j)\,\Pi_{j=1}^{D^{(1)}}\Gamma(d_j^{(1)})\ldots\Pi_{j=1}^{D^{(n)}}\Gamma(d_j^{(n)})}{\Pi_{j=1}^{A}\Gamma(a_j)\,\Pi_{j=1}^{B^{(1)}}\Gamma(b_j^{(1)})\ldots\Pi_{j=1}^{B^{(n)}}\Gamma(b_j^{(n)})}$$
$$\times \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty} \frac{\Pi_{j=1}^{A}\Gamma(a_j + \Sigma_{i=1}^{n} m_i\theta_j^{(i)})\,\Pi_{j=1}^{B^{(1)}}\Gamma(b_j^{(1)} + m_1\phi_j^{(1)})\ldots}{\Pi_{j=1}^{C}\Gamma(c_j + \Sigma_{i=1}^{n} m_i\Psi_j^{(i)})\,\Pi_{j=1}^{D^{(1)}}\Gamma(d_j^{(1)} + m_1\delta_j^{(1)})\ldots}$$
$$\times \frac{\Pi_{j=1}^{B^{(n)}}\Gamma(b_j^{(n)} + m_n\phi_j^{(n)}) z_1^{m_1}\ldots z_n^{m_n}}{\Pi_{j=1}^{D^{(n)}}\Gamma(d_j^{(n)} + m_n\delta_j^{(n)}) m_1!\ldots m_n!} \tag{4}$$

where the parameters $\boldsymbol{a}$, $\boldsymbol{b}$, $\theta^{(1)}, \ldots, \theta^{(n)}$, $\phi^{(1)}, \ldots$, etc, are vectors with the appropriate dimensions; for example,

$$\boldsymbol{a} = (a_1, \ldots, a_A) \qquad \text{and} \qquad \phi^{(k)} = (\phi_1^{(k)}, \ldots, \phi_{B^{(k)}}^{(k)})$$

are vectors of dimensionality A and $B^{(k)}$, respectively. The convergence conditions for these functions can be found in Slater (1966), Wright (1935, 1940), Appell and Kampé de Fériet (1926) and Srivastava and Daoust (1972), respectively. For $n=2$, the generalised Lauricella function defined in (4) is also referred to as the generalised Kampé de Fériet function.

The functions defined above are clearly interrelated; for example, (1) arises from (2) by setting $A_1 = \ldots = A_p = 1$ and $B_1 = \ldots = B_q = 1$ and multiplying the resulting expression by

$$\prod_{j=1}^{q} \Gamma(\beta_j)\left(\prod_{i=1}^{p} \Gamma(\alpha_i)\right)^{-1}.$$

Furthermore all the functions in (1)-(3) are contained within (4). This feature has often been exploited to yield new results for multiple variable hypergeometric functions from known results for single variable hypergeometric functions (see, e.g., the summation formula derived by Srivastava (1985b), equation (37)), and will play a similar role in this paper.

3. Reduction formula for an infinite series of $_4F_3(1)$

The half-space d-dimensional integral $f_{1/2}$ defined in the introduction can be evaluated to yield the following reduction of an infinite series of $_4F_3(1)$ to a $_3F_2(1)$:

$$\sum_{n=0}^{\infty} \frac{(1+\nu)_n}{n!2^n}\, {}_4F_3\left[\begin{matrix}\alpha/2,\, \alpha+\beta-d-\nu,\, (1+\alpha+\beta-d+n)/2,\, 1+(\alpha+\beta-d+n)/2; \\ (\alpha+\beta)/2,\, 1+(\alpha+\beta-d)/2,\, 1+\alpha+\beta-d+n;\end{matrix}\; 1\right]$$
$$= \frac{2^{1+\alpha+\beta-d}\Gamma(d+\nu-\alpha)\Gamma(\alpha/2+\beta/2)\Gamma(1+\alpha/2+\beta/2-d/2)}{\Gamma(\beta)\Gamma(d/2+\nu/2)\Gamma(1+\nu/2)}$$
$$\times\, {}_3F_2\left[\begin{matrix}(d+\nu-\alpha)/2,\, (\alpha+\beta-d-\nu)/2,\, (d-1)/2; \\ (1+\beta)/2,\, (d+\nu)/2;\end{matrix}\; 1\right] \tag{5}$$

where $\alpha < d+\nu < \alpha+\beta$ and $\nu > -1$.

The right-hand side of (5) provides a means of extending the domain of analyticity of the infinite series of $_4F_3(1)$. This can be achieved by applying to the $_3F_2(1)$ in (5) the 'two-term' relation (Slater 1966, p 114) given by

$$ {}_3F_2\left[\begin{matrix}a, b, c; \\ d, e;\end{matrix}\; 1\right] = \Gamma\left[\begin{matrix}d,\, e,\, s \\ a,\, s+b,\, s+c\end{matrix}\right] {}_3F_2\left[\begin{matrix}d-a,\, e-a,\, s; \\ s+b,\, s+c;\end{matrix}\; 1\right] \tag{6}$$

where $s \equiv d+e-a-b-c$ and

$$\Gamma\left[\begin{matrix}a_1, \ldots, a_m \\ b_1, \ldots, b_n\end{matrix}\right] \equiv \frac{\Pi_{i=1}^{m}\Gamma(a_i)}{\Pi_{i=1}^{n}\Gamma(b_i)}.$$

The proof of (5) follows by equating two evaluations of the integral $f_{1/2}$. In the first we obtain, after transforming to d-dimensional spherical coordinates, a double integral over θ and p, where the polar angle θ is that between the vector $\boldsymbol{p}$ and the unit vector $\mathbf{1}$, and the radial distance $p = |\boldsymbol{p}|$. On the other hand, we can also transform the integral by choosing distances x and $y \equiv p_d = \boldsymbol{p}\cdot\boldsymbol{1} \geqslant 0$, such that the norm $|\boldsymbol{p}+\boldsymbol{1}| = [x+(1+y)^2]^{1/2}$. Thus we obtain two equivalent representations of the integral $f_{1/2}$:

$$f_{1/2} = \frac{K_{d-1}}{2\pi}\int_0^{\pi/2} \mathrm{d}\theta \int_0^\infty \mathrm{d}p\, p^{d+\nu-\alpha-1}(\sin\theta)^{d-2}(\cos\theta)^\nu(1+2p\cos\theta+p^2)^{-\beta/2} \tag{7}$$

$$= \frac{K_{d-1}}{4\pi}\int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}x\, y^\nu x^{(d-3)/2}(x+y^2)^{-\alpha/2}[x+(1+y)^2]^{-\beta/2} \tag{8}$$

where the factor

$$K_d \equiv 2^{1-d}\pi^{-d/2}/\Gamma(d/2) \tag{9}$$

is the ratio of the surface area of a unit sphere in d dimensions to $(2\pi)^d$.

By the use of standard integrals (Gradshteyn and Ryzhik 1980, § 3.252, equation (10) and § 7.132, equation (6)), the integrals in (7) can easily be evaluated in the given order to yield the right-hand side of (5), to within a few factors of the gamma function. The evaluation of the integrals in (8) is somewhat more involved but the x integral can be expressed (Gradshteyn and Ryzhik 1980, § 3.197, equation (1)) in terms of a hypergeometric function

$${}_2F_1[\alpha/2, (d-1)/2; (\alpha+\beta)/2; 1-(1+y)^2/y^2].$$

Applying to this function Euler's transformation formula (Abramowitz and Stegun 1972, p 559)

$${}_2F_1[a, b; c; z] = (1-z)^{-a}{}_2F_1[a, c-b; c; z/(z-1)] \tag{10}$$

and substituting for the resulting hypergeometric function its series representation, term by term integration over the remaining variable y then results in an infinite series of ${}_2F_1(\frac{1}{2})$. This series can be written out as a doubly infinite series; inverting the order of summation and identifying the inner series with a ${}_4F_3(1)$ yields the left-hand side of (5) to within the factors of the gamma function, thus completing the proof.

4. Transformation of a ${}_4F_3(1)$ into a series of ${}_3F_2(1)$

The main general result of this section is the following new identity from which we deduce our symmetric summation formula:

$$\begin{aligned}
&{}_4F_3\left[{1, \beta/2, \alpha+\beta-d-\nu, (1+\alpha+\beta-d)/2; \atop (\alpha+\beta)/2, 1+(\alpha+\beta-d-\nu)/2, (1+\alpha+\beta-d-\nu)/2;} 1\right] \\
&\quad = -\frac{\alpha+\beta-d-\nu}{\alpha+\beta-d}\sum_{n=0}^{\infty}\frac{(-\nu)_n(-1)^n}{(1+d-\alpha-\beta)_n}\,{}_3F_2\left[{1+n, \alpha/2, \alpha+\beta-d-n; \atop (\alpha+\beta)/2, 1+(\alpha+\beta-d)/2;} 1\right] \\
&\quad + \left(\frac{\sin[\pi(\alpha+\beta-d-\nu)]}{\sin[\pi(\alpha+\beta-d)]} - 1\right) \\
&\quad \times \frac{\Gamma(1+\nu)\Gamma(1+\alpha+\beta-d-\nu)\Gamma(d+\nu-\alpha)}{\Gamma(\beta)\Gamma(1+\alpha+\beta-d)\Gamma(d/2+\nu/2)\Gamma(1+\nu/2)} \\
&\quad \times \Gamma(\alpha/2+\beta/2)\Gamma(1+\alpha/2+\beta/2-d/2)\,{}_3F_2\left[{(d+\nu-\alpha)/2, (\alpha+\beta-d-\nu)/2, (d-1)/2; \atop (1+\beta)/2, (d+\nu)/2;} 1\right] \\
&\quad + \frac{\pi^{1/2}B(d/2+\nu/2-\alpha/2, d/2-\beta/2)\Gamma(\alpha/2+\beta/2) \times\Gamma(1+\alpha/2+\beta/2-d/2-\nu/2)}{2^{\alpha+\beta-d-\nu-1}\Gamma(1/2+\alpha/2+\beta/2-d/2)\Gamma(\alpha/2-\nu/2)\Gamma(\beta/2)} \\
&\quad \times {}_3F_2\left[{-\nu/2, (d-1)/2, (d-\beta)/2; \atop (\alpha-\nu)/2, d+(\nu-\alpha-\beta)/2;} 1\right]
\end{aligned} \tag{11}$$

where $\alpha < d+\nu < \alpha+\beta$, $\nu > -1$ and $d > \beta$. The beta function $B(a, b)$ is given by $\Gamma(a)\Gamma(b)/\Gamma(a+b)$.

When the parameter ν in (11) is set equal to an integer ($m=0$, 1, 2, 3, ...) this infinite series terminates, thus providing us with a new transformation formula for a ${}_4F_3(1)$ into a finite number of ${}_3F_2(1)$. The case $m=0$ is particularly interesting in that it yields a new summation formula for a symmetric (with respect to interchange of α and β) combination of ${}_3F_2(1)$:

$$ {}_3F_2\left[{1, \beta/2, \alpha+\beta-d; \atop (\alpha+\beta)/2, 1+(\alpha+\beta-d)/2;} 1\right] + {}_3F_2\left[{1, \alpha/2, \alpha+\beta-d; \atop (\alpha+\beta)/2, 1+(\alpha+\beta-d)/2;} 1\right] $$

$$ = \frac{B(d/2-\alpha/2, d/2-\beta/2)\Gamma(\alpha/2+\beta/2-d/2)\Gamma(\alpha/2+\beta/2) \times\Gamma(1+\alpha/2+\beta/2-d/2)}{\Gamma(\alpha+\beta-d)\Gamma(\alpha/2)\Gamma(\beta/2)}. \tag{12} $$

This simple result would not be derivable from the two-term (see (6)) or three-term (Slater 1966, p 115) relations for a ${}_3F_2(1)$.

The summation formula (12) was obtained by making a special choice of parameters in a more general identity. This approach is quite common in the theory of hypergeometric functions and has often led to significant results. However, in some instances, due to what can be shown to be unjustified steps in the proofs of the more general identities, it has also led to apparently well established but false formulae. Examples of these formulae are equations (5) of Abiodun (1979) and (7) of Abiodun (1980), which have even been claimed by Srivastava (1985c, equation (1.6)) to be special cases of a more general result. Details of the false results and certain correct versions will appear elsewhere (Inayat-Hussain 1987b, Inayat-Hussain and Buckingham 1986b).

Most summation formulae, as is the case for our result (12), refer only to hypergeometric functions of arguments one or minus one (see Slater 1966, appendix III, Bailey 1935). Some other results are known, however, and we draw the reader's attention to a paper in which Gessel and Stanton (1982) have listed, for some generalised hypergeometric functions, a set of curious summation formulae which involve for example ${}_3F_2(\frac{3}{4})$, ${}_2F_1(\frac{1}{9})$, ${}_2F_1(\frac{8}{9})$, ... (We note in passing that equation (6.1) of the paper is clearly incorrect as can be seen by substituting $b=0$.)

The proof of the identity (11) is based on evaluating the integral f in two different ways, as in § 3. The calculations are carried out in the appendix with the results given in (A5) and (A9). By making use of our reduction formula (5) from the previous section, the infinite series of ${}_4F_3(1)$ in (A9) can be written as a single ${}_3F_2(1)$, and the resulting expression for f equated with (A5) to yield the required identity (11).

5. Conclusion

New results for hypergeometric functions of one or more variables have been discussed and provide an example indicating the existence of a new type of identity for symmetric combinations of hypergeometric functions.

It is curious that the integral $f_{1/2}$, which is merely the integral f over only the half-space, provides the very identity (5) required to simplify the evaluation (A9) of the whole integral f thus leading to the identity (11); without this simplification the analysis would have little significance. Indeed, while the relationship of $f_{1/2}$ to f is obvious, the way in which the infinite series of ${}_4F_3(1)$ arises in the expression (A9) for f is not, as is evident from inspection of the factors multiplying the series. We leave open the possibility that the clarification of this seemingly obscure connection might reveal yet another application of the integral f in providing new properties of hypergeometric functions.

The results presented in this paper, like many others in the theory of hypergeometric series, have the property that at first sight they seem obscure. This is particularly so in the case of (12), which is believed to be the first case of a summation formula for a pair of hypergeometric series, neither of which is individually summable. It would certainly be useful to have a criterion for establishing whether hypergeometric series are summable in terms of simple functions, for example, the gamma function. Even remarkably simple series can evade being summed, as illustrated by the example of the double series

$$\sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{m^r n^r (m+n)^r}.$$

Although this can be written as a particular Kampé de Fériet function with both arguments equal to -1, none of the known summation or reduction formulae applies. Subbarao and Sitaramachandrarao (1985) have remarked on the little success they obtained (even for the case when r is an even integer) in trying to sum the above series, in spite of much effort.

As well as the results reported, we have obtained (see Inayat-Hussain 1986) a new integral representation for a particular generalised Lauricella function of n variables. This representation has an advantage over other integral representations of the same function in that it readily yields a new reduction formula, a special case of which turns out to be the same as a special case of a known reduction formula for the multi-variable hypergeometric function studied by Niukkanen and Srivastava referred to in the introduction.

The q analogues (Exton 1983) of some of our results are under investigation and will be communicated in due course.

Acknowledgments

I am indebted to Professor M J Buckingham for many invaluable discussions and suggestions and for his help in the preparation of this paper. I also wish to thank John E Gottschalk for useful discussions and for access to his research material. The SMP (Inference Corp, USA) computer package has proved a powerful method for carrying out the numerical checks which have been so valuable in this work and I thank Kevin McIsaac and Paul C Abbott for their assistance in connection with this package.

Appendix. Evaluations of the integral f

The integral f (with $\pi-\theta$ the angle between $\mathbf{p}$ and $\mathbf{1}$) can be written in the following two different ways:

$$f(\alpha,\beta,\nu,d)=\frac{K_{d-1}}{2\pi}\int_0^\infty \mathrm{d}p\int_0^\pi \mathrm{d}\theta\frac{(\sin\theta)^{d-2}|\cos\theta|^\nu p^{d+\nu-\alpha-1}}{(1-2p\cos\theta+p^2)^{\beta/2}} \tag{A1}$$

$$=\frac{K_{d-1}}{4\pi}\int_{-\infty}^\infty \mathrm{d}y\int_0^\infty \mathrm{d}x\frac{|y|^\nu x^{(d-3)/2}}{(x+y^2)^{\alpha/2}[x+(1+y)^2]^{\beta/2}}. \tag{A2}$$

By expanding into a series in powers of $\cos\theta$ the denominator of the integrand in (A1), it is easy to show that

$$\int_0^\pi \mathrm{d}\theta\frac{(\sin\theta)^{d-2}|\cos\theta|^\nu}{[1-2p(1+p^2)^{-1}\cos\theta]^{\beta/2}}$$

$$=\frac{\Gamma(d/2-1/2)\Gamma(1/2+\nu/2)}{\Gamma(d/2+\nu/2)}\,{}_3F_2\left[{(1+\nu)/2,\,\beta/4,\,1/2+\beta/4;\atop (d+\nu)/2,\,1/2;}\;4p^2/(1+p^2)^2\right] \tag{A3}$$

substitution of which into (A1) followed by a term by term integration over the remaining variable p yields

$$f(\alpha,\beta,\nu,d)=\frac{\Gamma(1/2+\nu/2)\Gamma(d/2+\nu/2-\alpha/2)\Gamma(\alpha/2+\beta/2-d/2-\nu/2)}{2^d\pi^{(1+d)/2}\Gamma(\beta/2)\Gamma(d/2+\nu/2)}$$

$$\times\,{}_3F_2\left[{(1+\nu)/2,\,(d+\nu-\alpha)/2,\,(\alpha+\beta-d-\nu)/2;\atop (d+\nu)/2,\,1/2;}\;1\right]. \tag{A4}$$

The ${}_3F_2(1)$ in (A4) can be transformed via the two-term relation (6) into another ${}_3F_2(1)$, thus completing one evaluation of the integral f:

$$f(\alpha,\beta,\nu,d)=\frac{\Gamma(\alpha/2+\beta/2-d/2-\nu/2)B(d/2+\nu/2-\alpha/2,\,d/2-\beta/2)}{2^d\pi^{d/2}\Gamma(\alpha/2-\nu/2)\Gamma(\beta/2)}$$

$$\times\,{}_3F_2\left[{-\nu/2,\,(d-1)/2,\,(d-\beta)/2;\atop (\alpha-\nu)/2,\,d+(\nu-\alpha-\beta)/2;}\;1\right]. \tag{A5}$$

It is interesting to note that for $\nu=2m$ $(m=0,1,2,3,\ldots)$ the ${}_3F_2(1)$ in (A5) terminates whereas the ${}_3F_2(1)$ in (A4) is non-terminating. This case provides an illustration of the usefulness of the two-term relation.

Turning now to the second form (A2), we observe that the x integral can be evaluated (Gradshteyn and Ryzhik 1980, § 3.197, equation (1)) in terms of a ${}_2F_1$ hypergeometric function. A twofold application of Euler's transformation (10) to this ${}_2F_1$ followed by a change of variable $(y\to -y'/2-\frac{1}{2})$ in the remaining integral results in

$$f(\alpha,\beta,\nu,d)$$

$$=\frac{2^{\alpha+\beta-2d-\nu}\Gamma(1/2+\alpha/2+\beta/2-d/2)}{\pi^{(1+d)/2}\Gamma(\alpha/2+\beta/2)}$$

$$\times\left(\int_0^\infty \mathrm{d}y\frac{{}_2F_1[\beta/2,1/2+\alpha/2+\beta/2-d/2;\alpha/2+\beta/2;1-(1-y)^2/(1+y)^2]}{(1+y)^{1+\alpha+\beta-d-\nu}}\right.$$

$$\left.+\int_0^\infty \mathrm{d}y\frac{|1-y|^\nu{}_2F_1[\alpha/2,1/2+\alpha/2+\beta/2-d/2;\alpha/2+\beta/2;1-(1-y)^2/(1+y)^2]}{(1+y)^{1+\alpha+\beta-d}}\right). \tag{A6}$$

The first integral on the right of (A6), which we will denote by A, can be evaluated by substituting the series representation for the ${}_2F_1$ followed by a term by term integration to give

$$A=(\alpha+\beta-d-\nu)^{-1}{}_4F_3\left[{1,\,\beta/2,\,\alpha+\beta-d-\nu,\,(1+\alpha+\beta-d)/2;\atop (\alpha+\beta)/2,\,1+(\alpha+\beta-d-\nu)/2,\,(1+\alpha+\beta-d-\nu)/2;}\;1\right]. \tag{A7}$$

To evaluate the second integral on the right of (A6), which we will denote by B, we first split the integral into two parts from 0 to 1 and from 1 to infinity. The individual integrals can now be evaluated in a similar way as for A to yield an infinite series over a combination of two hypergeometric functions:

$$ {}_2F_1[1+\alpha+\beta-d+2n, 1+n; 2+\nu+n; -1] $$

and

$$ {}_2F_1[1+\alpha+\beta-d+2n, \alpha+\beta-d-\nu+n; 1+\alpha+\beta-d+n; -1]. $$

By the application of another well known linear transformation formula (Abramowitz and Stegun 1972, equation (15.3.7)) the latter hypergeometric function can be replaced by two such functions, one of which combines with the former hypergeometric function. These manipulations followed by another use of (10) lead to

$$ B=(\alpha+\beta-d)^{-1}\sum_{n=0}^{\infty}\frac{(\alpha/2)_n(\alpha+\beta-d)_n\,{}_2F_1[1+n,-\nu;1+d-\alpha-\beta-n;-1]}{(\alpha/2+\beta/2)_n(1+\alpha/2+\beta/2-d/2)_n} $$
$$ +\frac{2^{d-1-\alpha-\beta}\Gamma(1+\nu)\Gamma(\alpha+\beta-d-\nu)}{\Gamma(1+\alpha+\beta-d)}\left(1-\frac{\sin[\pi(\alpha+\beta-d-\nu)]}{\sin[\pi(\alpha+\beta-d)]}\right) $$
$$ \times\sum_{n=0}^{\infty}\frac{(\alpha/2)_n(1/2+\alpha/2+\beta/2-d/2)_n(\alpha+\beta-d-\nu)_n}{(\alpha/2+\beta/2)_n(1+\alpha+\beta-d)_n n!} $$
$$ \times\,{}_2F_1\left[{1+\alpha+\beta-d+2n,\ 1+\nu;\atop 1+\alpha+\beta-d+n;}\ 1/2\right]. \qquad \text{(A8)} $$

The doubly infinite series in both terms of (A8) can be inverted to yield an infinite series of ${}_3F_2(1)$ and an infinite series of ${}_4F_3(1)$. Finally, combining the resulting expression for B with the expression for A as given in (A7) provides us with the required second evaluation of the integral f:

$$ f(\alpha,\beta,\nu,d)=\frac{2^{-d}\pi^{-d/2}\Gamma(1/2+\alpha/2+\beta/2-d/2)\Gamma(\alpha+\beta-d-\nu)}{\Gamma(\alpha/2+\beta/2)\Gamma(1/2+\alpha/2+\beta/2-d/2-\nu/2)\times\Gamma(1+\alpha/2+\beta/2-d/2-\nu/2)} $$
$$ \times\Bigg[{}_4F_3\left[{1,\ \beta/2,\ \alpha+\beta-d-\nu,\ (1+\alpha+\beta-d)/2;\atop (\alpha+\beta)/2,\ 1+(\alpha+\beta-d-\nu)/2,\ (1+\alpha+\beta-d-\nu)/2;}\ 1\right] $$
$$ +\frac{\alpha+\beta-d-\nu}{\alpha+\beta-d}\sum_{n=0}^{\infty}\frac{(-\nu)_n(-1)^n}{(1+d-\alpha-\beta)_n}\,{}_3F_2\left[{1+n,\ \alpha/2,\ \alpha+\beta-d-n;\atop (\alpha+\beta)/2,\ 1+(\alpha+\beta-d)/2;}\ 1\right] $$
$$ +\frac{2^{d-1-\alpha-\beta}\Gamma(1+\nu)\Gamma(1+\alpha+\beta-d-\nu)}{\Gamma(1+\alpha+\beta-d)}\left(1-\frac{\sin[\pi(\alpha+\beta-d-\nu)]}{\sin[\pi(\alpha+\beta-d)]}\right) $$
$$ \times\sum_{n=0}^{\infty}\frac{(1+\nu)_n}{n!2^n}\,{}_4F_3\left[{\alpha/2,\ \alpha+\beta-d-\nu,\ (1+\alpha+\beta-d+n)/2,\ 1+(\alpha+\beta-d+n)/2;\atop (\alpha+\beta)/2,\ 1+(\alpha+\beta-d)/2,\ 1+\alpha+\beta-d+n;}\ 1\right]\Bigg]. \qquad \text{(A9)} $$

References

Abiodun R F A 1979 *J. Math. Anal. Appl.* **70** 114-9

—— 1980 *Nederl. Akad. Wetensch. Indag. Math.* **42** 1-11

Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (New York: Dover)

Appell P and Kampé de Fériet J 1926 *Fonctions hypergéométriques et hypersphériques, polynomes d'Hermite* (Paris: Gauthier-Villars)
Bailey W N 1935 *Generalised Hypergeometric Series* (Cambridge: Cambridge University Press)
Erdélyi A, Magnus W, Oberhettinger F and Tricomi F G 1953 *Higher Transcendental Functions* vols 1, 2, 3 (New York: McGraw-Hill)
Exton H 1978 *Handbook of Hypergeometric Integrals: Theory, Applications, Tables, Computer Programs* (New York: Halsted/Wiley)
—— 1983 *q-Hypergeometric Functions and Applications* (New York: Halsted/Wiley)
Gessel I and Stanton D 1982 *SIAM J. Math. Anal.* **13** 295-308
Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals, Series and Products* (New York: Academic)
Inayat-Hussain A A 1986 *PhD Thesis* in preparation
—— 1987a *J. Phys. A: Math. Gen.* **20** 4119-28
—— 1987b *Aust. Math. Soc. Gazette* to be published
Inayat-Hussain A A and Buckingham M J 1986a *Proc. 16th IUPAP Int. Conf. on Thermodynamics and Statistical Mechanics* to be published
——1986b to be published
Kampé de Fériet J 1921 *C.R. Acad. Sci., Paris* **173** 401-4
Lauricella G 1893 *Rend. Circ. Mat. Palermo* **7** 111-58
Ma S K 1976 *Modern Theory of Critical Phenomena* (New York: Benjamin)
Niukkanen A W 1983 *J. Phys. A: Math. Gen.* **16** 1813-25
—— 1984 *J. Phys. A: Math. Gen.* **17** L731-6
Slater L J 1966 *Generalised Hypergeometric Functions* (Cambridge: Cambridge University Press)
Srivastava H M 1985a *J. Phys. A: Math. Gen.* **18** L227-34
—— 1985b *J. Phys. A: Math. Gen.* **18** 3079-85
—— 1985c *J. Math. Anal. Appl.* **107** 498-508
Srivastava H M and Daoust M C 1969 *Nederl. Akad. Wetensch. Indag. Math.* **31** 449-57
—— 1972 *Math. Nachr.* **53** 151-9
Srivastava H M and Goyal S P 1985 *J. Math. Anal. Appl.* **112** 641-51
Srivastava H M, Gupta K C and Goyal S P 1982 *The H-functions of One and Two Variables with Applications* (New Delhi: South Asian)
Srivastava H M and Karlsson P W 1985 *Multiple Gaussian Hypergeometric Series* (New York: Halsted/Wiley)
Subbarao M V and Sitaramachandrarao R 1985 *Pacific J. Math.* **119** 245-55
Wright E M 1935 *J. London Math. Soc.* **10** 286-93
—— 1940 *Proc. London Math. Soc. (2)* **46** 389-408